Q1 QAM Key Equations (Cos-Sin Form)

Sunday, August 05, 2012 9:20 PM

Let $v(t) = x_{QAM}(t)\sqrt{2}\sin(2\pi f_c t)$

$$= m_1(t)\underbrace{2\cos(2\pi f_c t)\sin(2\pi f_c t)} + m_2(t)\underbrace{2\sin(2\pi f_c t)\sin(2\pi f_c t)}$$

$$2\cos x \sin x = \cancel{2}\left(\frac{e^{jx}+e^{-jx}}{\cancel{2}}\right)\left(\frac{e^{jx}-e^{-jx}}{2j}\right) = \frac{1}{2j}\left(e^{j2x}-e^{-j2x}\right) = \sin(2x)$$

$$\sin^2 x = \left(\frac{e^{jx}-e^{-jx}}{2j}\right)^2 = \frac{e^{2jx}+e^{-2jx}-2}{-4} = \frac{1}{2}\left(1-\cos(2x)\right)$$

$$= m_1(t)\sin(2\pi(2f_c)t) + m_2(t)\left(1-\cos(2\pi(2f_c)t)\right)$$

$$= m_2(t) + \underbrace{m_1(t)\sin(2\pi(2f_c)t)}_{\substack{\text{the spectrum}\\ \text{is centered}\\ \text{around } \pm 2f_c}} - \underbrace{m_2(t)\cos(2\pi(2f_c)t)}_{\substack{\text{the spectrum}\\ \text{is centered}\\ \text{around } \pm 2f_c}}$$

$$\mathrm{LPF}\{v(t)\} \underset{\uparrow}{=} m_2(t) + 0 + 0 = m_2(t)$$

Assumption:

If $m_1(t)$ is band limited to B_1,

$m_2(t)$ is band limited to B_2,

we need $f_c > B_1$ and $f_c > B_2$

Q2 QAM: Envelope-and-Phase Form

Sunday, November 08, 2015 12:25 AM

(a) At a particular time t (or over time interval $t_1 \le t \le t_2$ where $m_1(t)$ and $m_2(t)$ are constant),

suppose $m_1(t) \equiv m_1$ and $m_2(t) \equiv m_2$.

Then $x_{QAM}(t) = m_1\sqrt{2}\cos(2\pi f_c t) + m_2\sqrt{2}\sin(2\pi f_c t)$

$= m_1\sqrt{2}\cos(2\pi f_c t) + m_2\sqrt{2}\cos(2\pi f_c t - 90^\circ)$

$\Updownarrow \qquad \Updownarrow$

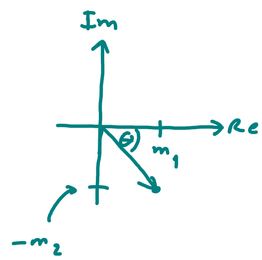
$m_1\sqrt{2}\angle 0^\circ + m_2\sqrt{2}\angle -90^\circ$

$=$

$\sqrt{2}(m_1 - jm_2)$

$= \sqrt{2}E\angle\theta$ ← polar form

using the phasor representation to combine the two sinusiods

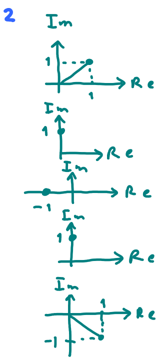


$\Updownarrow$

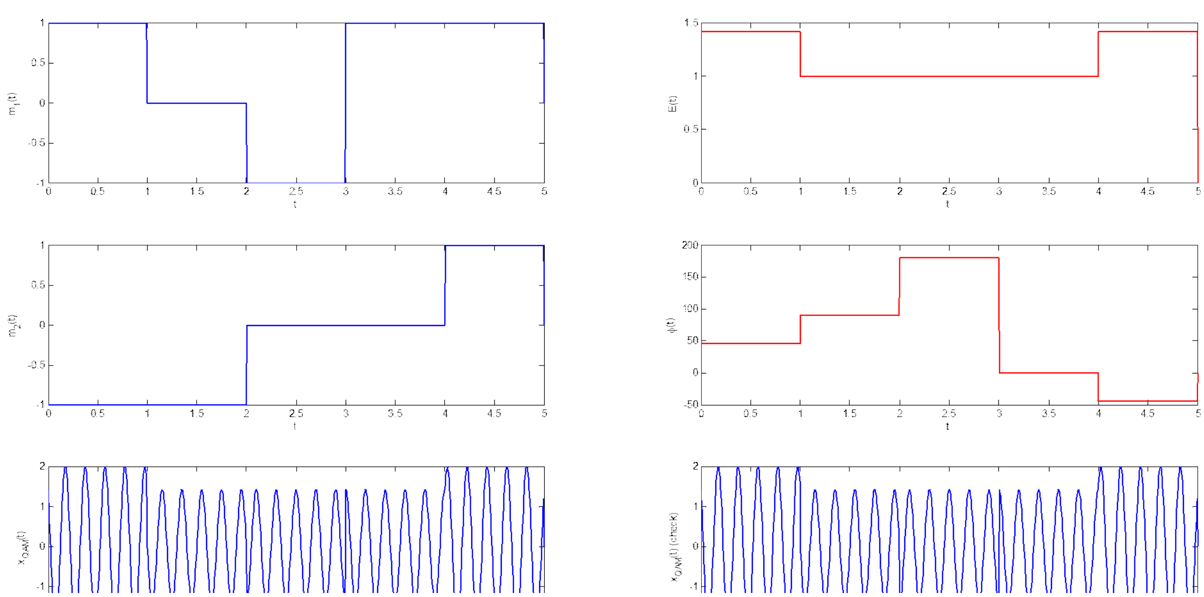
$= \sqrt{2}E\cos(2\pi f_c t + \theta)$

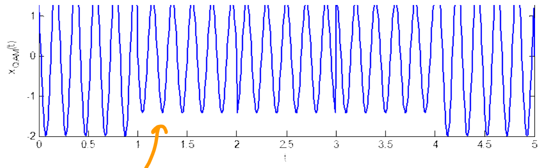
convert this to polar form. $E\angle\phi$

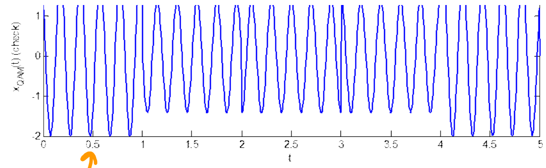
m_1	m_2	$m_1 - jm_2$		E	ϕ
1	-1	$1+j$		$\sqrt{1^2+1^2} = \sqrt{2}$	45°
0	-1	j		1	90°
-1	0	-1		1	180°
1	0	1		1	0°
1	1	$1-j$		$\sqrt{1^2+(-1)^2} = \sqrt{2}$	-45°



Note that once we get the value of $m_1 - jm_2$, the conversion to polar form can be done easily inside your calculator.
Here, the conversion is visually obvious; so, in fact, we can do the conversion without the help of a calculator.







$x_{QAM}(t) = \sqrt{2}\, m_1(t) \cos(2\pi f_c t) + \sqrt{2}\, m_2(t) \sin(2\pi f_c t)$

$x_{QAM}(t) = \sqrt{2}\, E(t) \cos(2\pi f_c t + \phi(t))$

They should look exactly the same.

(b) As hinted, we apply the trig. identity

$$\cos A \cos B + \sin A \sin B = \cos(A-B)$$

As usual, this trig. identity can be proved via the Euler's formula:

$$\cos A \cos B = \left(\frac{e^{jA} + e^{-jA}}{2}\right) \times \left(\frac{e^{jB} + e^{-jB}}{2}\right)$$

$$= \frac{1}{4}\left(e^{j(A+B)} + e^{j(A-B)} + e^{-j(A-B)} + e^{-j(A+B)}\right)$$

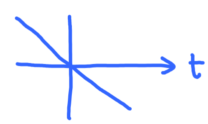
$$\sin A \sin B = \left(\frac{e^{jA} - e^{-jA}}{2j}\right) \times \left(\frac{e^{jB} - e^{-jB}}{2j}\right)$$

$$= -\frac{1}{4}\left(e^{j(A+B)} - e^{j(A-B)} - e^{-j(A-B)} + e^{-j(A+B)}\right)$$

$$\cos A \cos B + \sin A \sin B = \frac{1}{\cancel{4}_{2}} \times \cancel{2}\left(e^{j(A-B)} + e^{-j(A-B)}\right) = \cos(A-B)$$

$$x_{QAM}(t) = \sqrt{2}\cos(2\pi B t)\cos(2\pi f_c t) + \sqrt{2}\sin(2\pi B t)\sin(2\pi f_c t)$$

$$= \underbrace{\sqrt{2}}_{E(t) = \sqrt{2}} \cos(2\pi f_c t \underbrace{- 2\pi B t}_{= \phi(t)\text{? No!}})$$

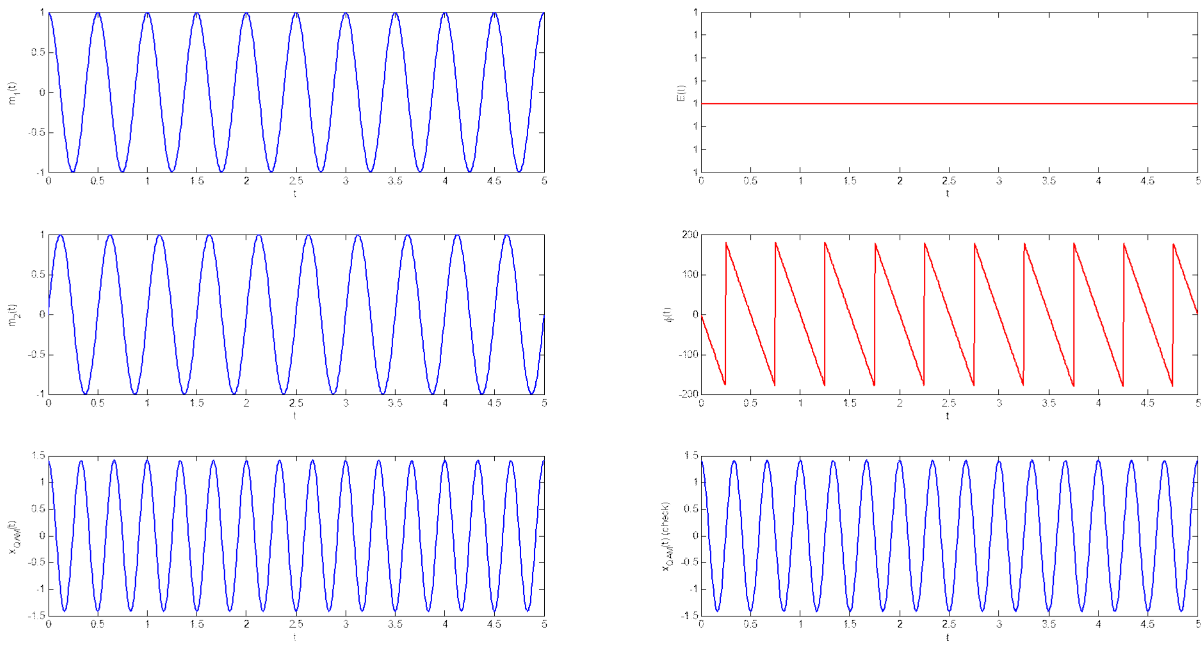


We can't stop here because the question needs $\phi(t) \in [-180°, 180°]$.

It is clear that when t is large, "$-2\pi B t$" will exceed $-180°$. We may refer to "$-2\pi B t$" as the "unwrapped phase". Adding/subtracting appropriate multiple of $360°$ will bring "$-2\pi B t$" into the $[-180°, 180°]$ range. Mathematically, this could be done via

$$\phi(t) = \left(\left(-2\pi B t + 180°\right) \bmod 360°\right) - 180°.$$

We may refer to this as the "wrapped phase".



(c) We can apply the command "abs" and "angle" to $m_1(t) - j m_2(t)$ to find $E(t)$ and $\phi(t)$ respectively.

Alternatively, we can find $E(t)$ from

$$E(t) = \sqrt{m_1^2(t) + m_2^2(t)} = \sqrt{\cos^2(2\pi B t) + (2\sin(2\pi B t))^2} = \sqrt{\cos^2 A + 4\sin^2 A}$$

$A = 2\pi B t$

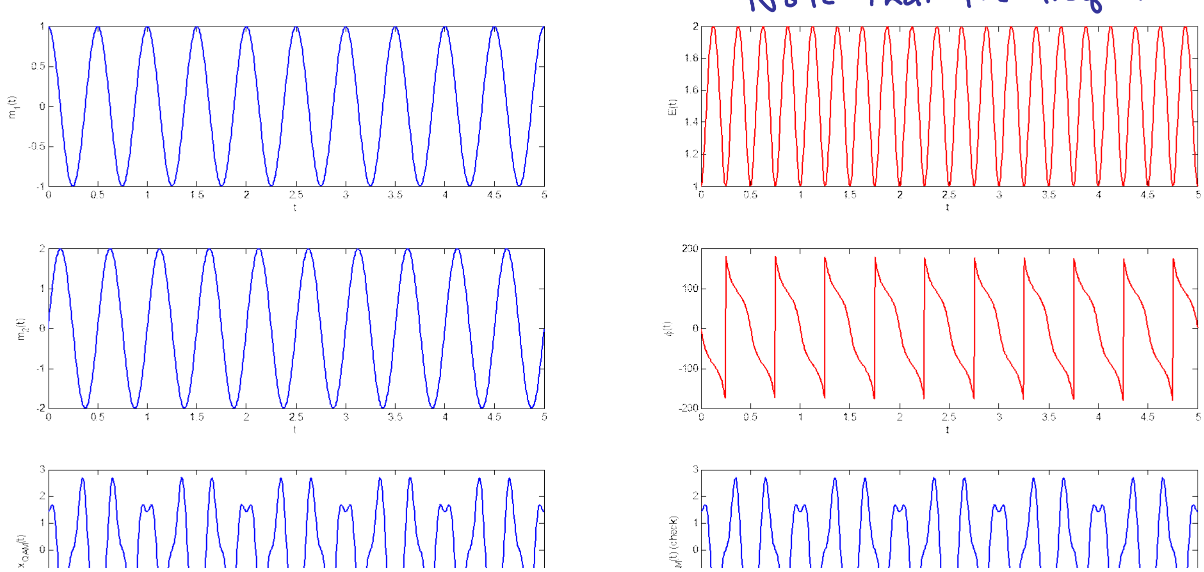
$$= \sqrt{\cos^2 A + 4(1-\cos^2 A)} = \sqrt{4 - 3\cos^2 A}$$

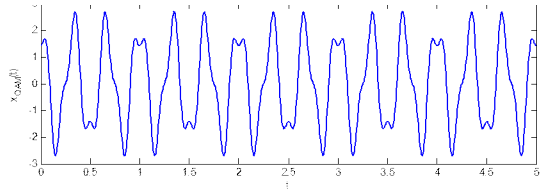
$$= \sqrt{4 - 3\left(\frac{1}{2}(1+\cos(2A))\right)} = \sqrt{\frac{5}{2} - \frac{3}{2}\cos(2A)} = \sqrt{\frac{5}{2} - \frac{3}{2}\cos(2\pi(2B)t)}$$

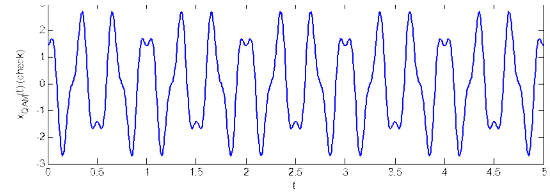
The command atan2$(m_1(t), -m_2(t))$ can also be used to find $\phi(t)$.

Caution: Because the question want $\phi(t) \in (-180°, 180°]$, don't forget to convert the unit from "radians" to "degrees".

Note that the freq. is doubled.




$x_{QAM}(t)$
t
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
-3
-2
-1
0
1
2


$x_{QAM}(t)$ (check)
t
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
-3
-2
-1
0
1
2

Q3 QAM with Phase and Frequency Offset

Sunday, November 8, 2015 8:46 PM

$$x_{QAM}(t) = m_1(t)\sqrt{2}\cos(\omega_c t) + m_2(t)\sin(\omega_c t)$$

Recall the product-to-sum formula:

$$\cos A \cos B = \frac{1}{2}\left(\cos(A+B) + \cos(A-B)\right).$$

In this question, we will need two more formulas involving products of sine and cosine functions.

Again, we will use the Euler's identity:

$$\sin A = \frac{1}{2j}\left(e^{jA} - e^{-jA}\right)$$

$$\cos B = \frac{1}{2}\left(e^{jB} + e^{-jB}\right)$$

Hence,

$$\sin A \cos B = \frac{1}{4j}\left(e^{jA} - e^{-jA}\right)\left(e^{jB} + e^{-jB}\right) = \frac{1}{4j}\left(e^{j(A+B)} - e^{-j(A+B)} + e^{j(A-B)} - e^{-j(A-B)}\right)$$

$$= \frac{1}{4j}\left(2j\sin(A+B) + 2j\sin(A-B)\right) = \frac{1}{2}\sin(A+B) + \frac{1}{2}\sin(A-B),$$

and

$$\sin A \sin B = \frac{1}{(2j)^2}\left(e^{jA} - e^{-jA}\right)\left(e^{jB} - e^{-jB}\right) = \frac{1}{-4}\left(e^{j(A+B)} + e^{-j(A+B)} - e^{j(A-B)} - e^{-j(A-B)}\right)$$

$$= -\frac{1}{4}\left(2\cos(A+B) - 2\cos(A-B)\right) = \frac{1}{2}\left(\cos(A-B) - \cos(A+B)\right)$$

(a) Let $v_1(t) = x_{QAM}(t)\sqrt{2}\cos\left((\omega_c + \Delta\omega)t + \delta\right)$

and $\hat{m}_1(t) = \text{LPF}\{v_1(t)\}$.

Then, by the product-to-sum formulas,

$$v_1(t) = m_1(t)\cos\left(\underbrace{(2\omega_c + \Delta\omega)t + \delta}_{\text{LPF} \to 0}\right) + m_1(t)\cos\left((\Delta\omega)t + \delta\right)$$

$$+ m_2(t)\sin\left(\underbrace{(2\omega_c + \Delta\omega)t + \delta}_{\text{LPF} \to 0}\right) + m_2(t)\sin\left(-(\Delta\omega)t - \delta\right)$$

$$\hat{m}_1(t) = m_1(t)\cos\left((\Delta\omega)t + \delta\right) - m_2(t)\sin\left((\Delta\omega)t + \delta\right)$$

(b) Let $v_2(t) = x_{QAM}(t)\sqrt{2}\sin\left((\omega_c + \Delta\omega)t + \delta\right)$

and $\hat{m}_2(t) = \text{LPF}\{v_2(t)\}$.

Then, by the product-to-sum formulas,

$$v_2(t) = m_1(t)\sin\left(\underbrace{(2\omega_c + \Delta\omega)t + \delta}_{\text{LPF} \to 0}\right) + m_1(t)\sin\left((\Delta\omega)t + \delta\right)$$

$$+ m_2(t)\cos\left((\Delta\omega)t + \delta\right) - m_2(t)\cos\left(\underbrace{(2\omega_c + \Delta\omega)t + \delta}_{\text{LPF} \to 0}\right)$$

$$\hat{m}_2(t) = m_1(t)\sin\left((\Delta\omega)t + \delta\right) + m_2(t)\cos\left((\Delta\omega)t + \delta\right)$$

Q4 AM FM PM signals

Sunday, November 8, 2015 2:30 PM

(a) During time $t=0$ to $t=10^{-3}$, there are 5 cycles of $\cos(2\pi f_c t + \phi)$.

Therefore, its frequency is $f_c = \frac{5}{10^{-3}} = 5$ kHz.

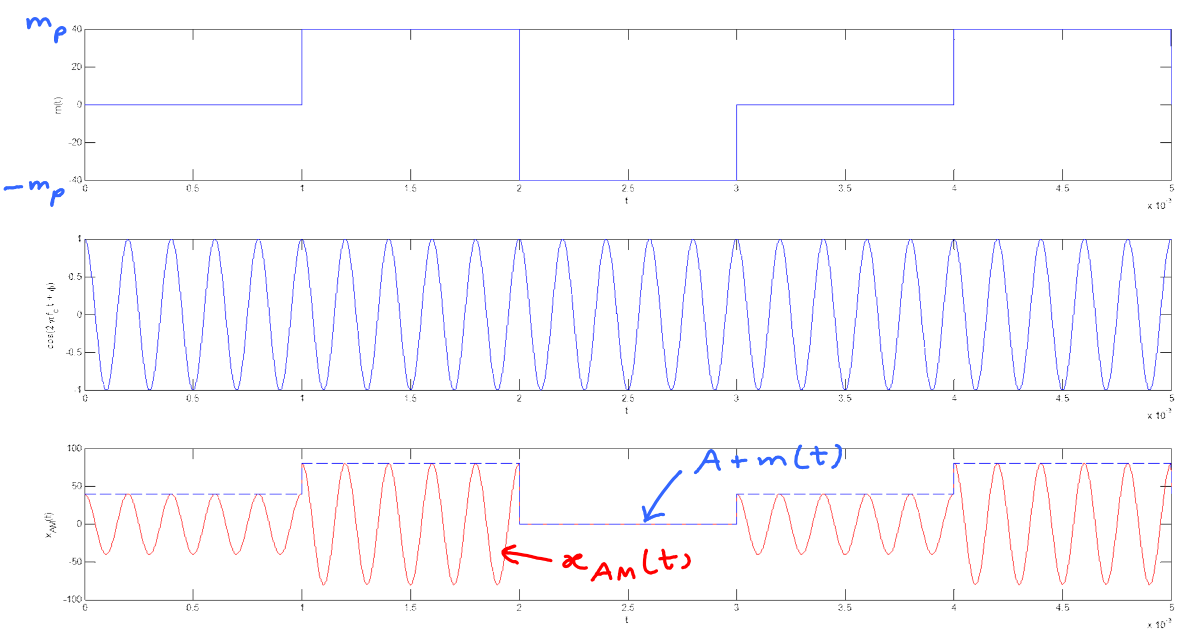
(b) (b.i)

$x_{AM}(t) = (A + m(t))\cos(2\pi f_c t + \phi)$.

Here, the value of A is not given. However, we can find it from the modulation index value. Recall that $\mu = \frac{A}{m_p}$. Therefore, $A = \mu m_p$.

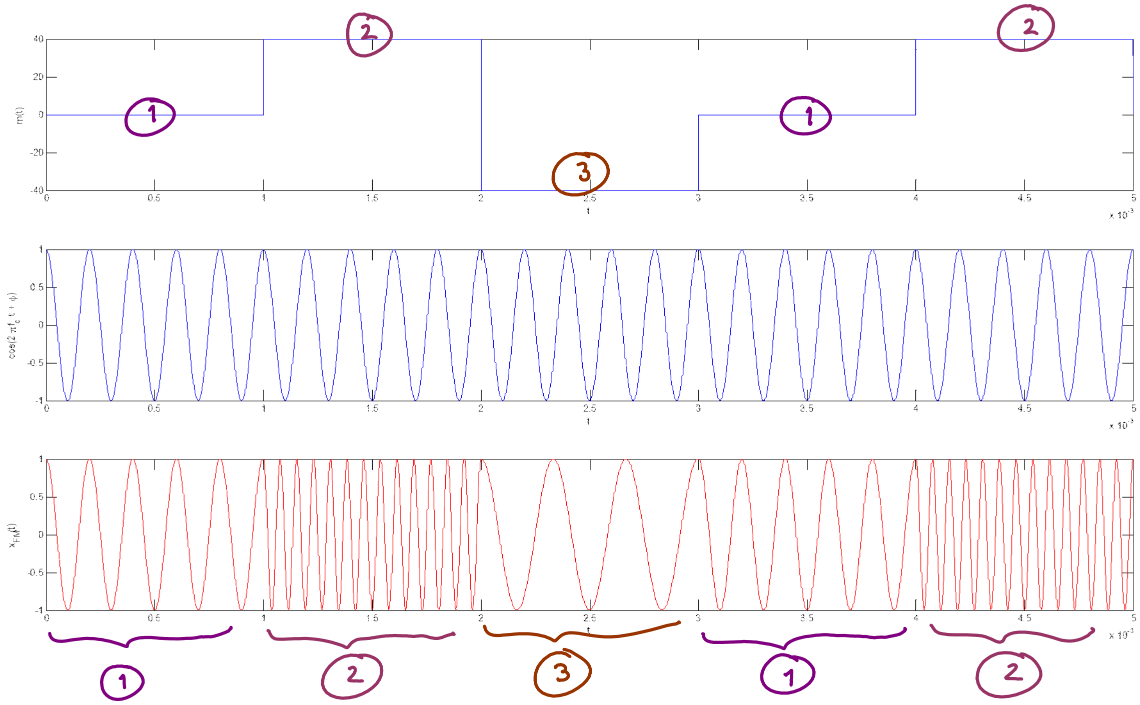
In this problem, $\mu = 100\% = 1$ and $m_p = 40$. Therefore, $A = 40$. Note that even when you can not read $m_p = 40$ from the graph of A, the important information here is that, with $\mu = 100\%$, we have $A = m_p$.

With $\min_t m(t) = -m_p$, we know that $A + m(t)$ will be 0 when $m(t)$ is having its minimum value.



(b.ii)

For FM, we use the fact that the instantaneous freq. of $x_{FM}(t)$ should be $f(t) = f_c + k_f m(t)$.



Case 1 : When $m(t) = 0$, we should have $f(t) = f_c + 0 = f_c$.

Case 2 : When $m(t)$ is at its "high" value, the inst. freq. $f(t)$ of $x_{FM}(t)$ should also be at its "high" value.

The "high" value of $m(t)$ is > 0. So, the corresponding $f(t) = f_c + k_f m(t)$ should be higher than the carrier freq f_c.

Case 3 : When $m(t)$ is at its "low" value, the inst. freq. $f(t)$ of $x_{FM}(t)$ should also be at it "low" value.

(b.iii)

For PM, note that $m(t)$ is piecewise-constant. Its values jump at various places but there is no place that $m(t)$ change gradually. The derivative of $m(t)$ is 0 almost everywhere except at the jump locations. Recall that the inst. freq of $x_{PM}(t)$ is

$$f(t) = f_c + \frac{k_p}{2\pi} \dot{m}(t).$$

Therefore, $f(t) = f_c$ almost everywhere.

At each of the jump locations, $x_{PM}(t)$ should have sudden phase change. Suppose the $m(t)$ increases by Δm, then the phase of $x_{PM}(t)$ at that location should suddenly advance by $\Delta \phi = k_p \Delta m$. Here, $k_p = \frac{\pi}{m}$. So, $\Delta \phi = \pi \times \frac{\Delta m}{m}$.

$x_{PM}(t)$ at that location should suddenly advance by $\Delta\phi = k_p \Delta m$.

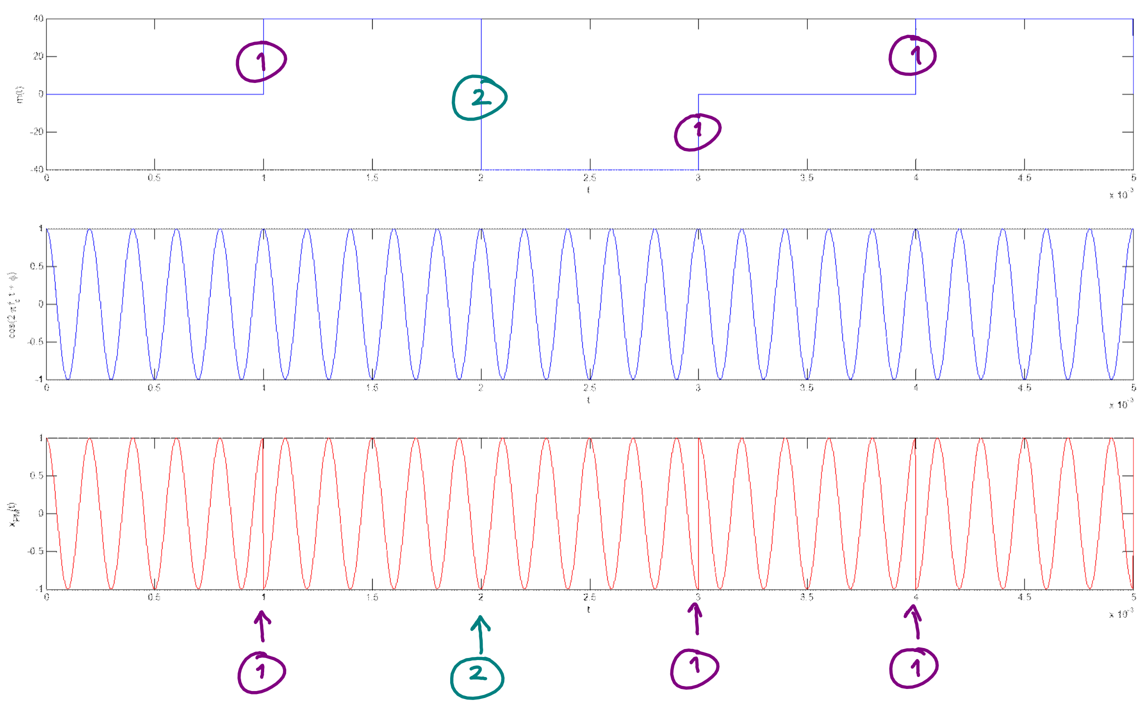
Here, $k_p = \frac{\pi}{m_p}$. So, $\Delta\phi = \pi \times \frac{\Delta m}{m_p}$.

Jump location

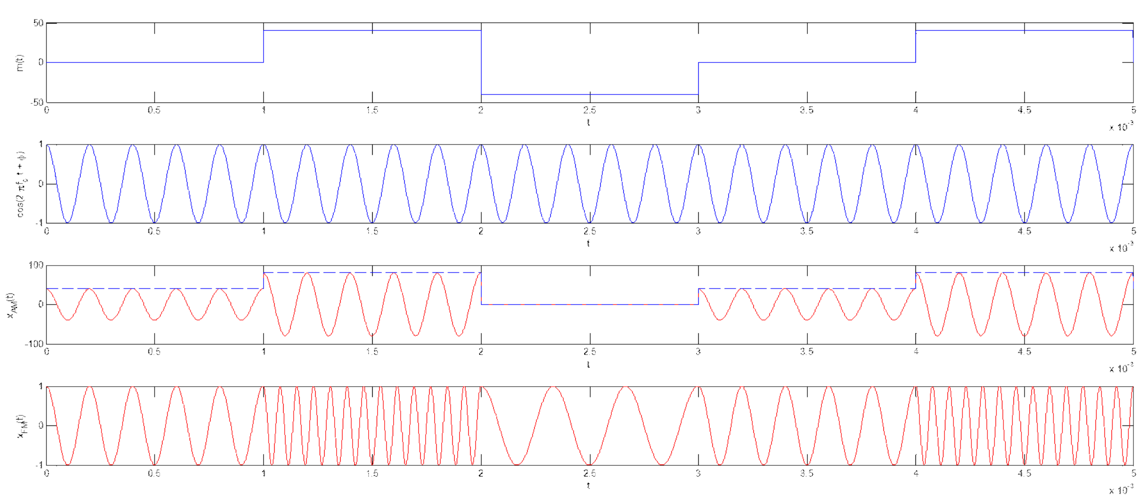
t [ms]	Δm	$\Delta\phi$
1	$+m_p$	π
2	$-2m_p$	-2π
3	$+m_p$	π
4	$+m_p$	π

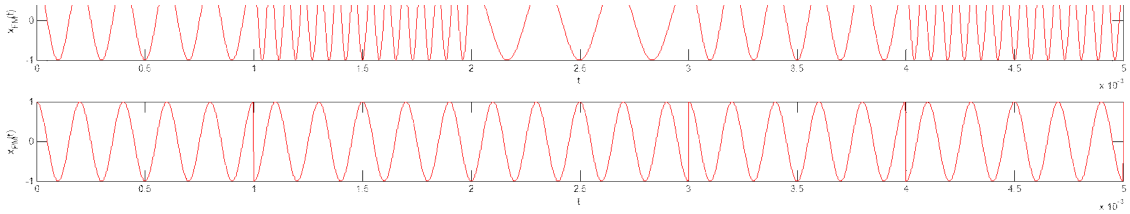
Case 2: When a sinusoid is advanced by "-2π", we simply get the same waveform: $\cos(\beta - 2\pi) = \cos(\beta)$.

Case 1: When a sinusoid is advanced by π, we simply get its "$-$": $\cos(\beta + \pi) = -\cos(\beta)$.



(c) Here, in part (b), the "sketches" are already drawn by MATLAB. Here, we simply put all of them together:





Q5 FM BW

Sunday, November 08, 2015 10:02 PM

First, observe that $m(t)$ takes only 3 values: $-40, 0, 40$.

(a) In class, we discuss the fact that during the time duration T_s that $x_{FM}(t)$ has instantaneous freq. $= f_0$, its Fourier transform contribution is a sinc function centered at $\pm f_0$.

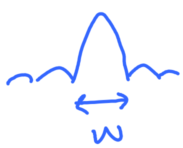
For FM, we know that $f(t) = f_c + k_f m(t)$. Therefore, here, the inst. freq. of $x_{FM}(t)$ will take only 3 values. Plugging-in the possible values of $m(t)$ (from the lowest one to the highest one), we get

$$f_1 = f_c + k_f(-40) = 5\times10^3 + 75(-40) = 5000 - 3000 = 2 \text{ kHz}$$

$$f_2 = f_c + k_f(0) = f_c = 5 \text{ kHz}$$

$$f_3 = f_c + k_f(40) = 5\times10^3 + 75(40) = 5000 + 3000 = 8 \text{ kHz}$$

(b) The "width" of each of the sinc pulse in the freq. domain is $W = \frac{2}{T_s}$.



Here, from the plot of $m(t)$, we have $T_s = 1$ ms.

Therefore, $W = \dfrac{2}{1\times10^{-3}} = 2 \text{ kHz}$.

(c) $$BW = \frac{1}{T_s} + (\underbrace{f_3}_{f_{max}} - \underbrace{f_1}_{f_{min}}) + \frac{1}{T_s} = (8-2) + 2 = 8 \text{ kHz}$$

Q6 More properties of FT

Monday, November 09, 2015 10:40 AM

Consider $x(t) \xrightarrow{\mathcal{F}} X(f)$.

(a) Let $y(t) = x^*(t)$. We want to find $Y(f)$.

First, recall that $X(f) = \int_{-\infty}^{\infty} x(t)\, e^{j2\pi f t}\, dt$.

Hence, $Y(f) = \int_{-\infty}^{\infty} x^*(t)\, e^{j2\pi f t}\, dt = \Big(\underbrace{\int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt}_{X(-f)} \Big)^* = \left(X(-f)\right)^* = X^*(-f)$

(b) Let $y(t) = \mathrm{Re}\{x(t)\}$.

From the hint, we first note that $x(t) + x^*(t) = 2\,\mathrm{Re}\{x(t)\}$.

Hence, $y(t) = \mathrm{Re}\{x(t)\} = \frac{1}{2}\left(x(t) + x^*(t)\right)$ and

$$Y(f) = \frac{1}{2}\left(X(f) + \mathcal{F}\{x^*(t)\}\right) = \frac{1}{2}\left(X(f) + X^*(-f)\right)$$

From part (a)

Remarks: (1) The expression for $Y(f)$ above is similar to $\mathrm{Re}\{X(f)\}$ but they are not the same.

Compare:

$$\mathrm{Re}\{X(f)\} = \frac{1}{2}\left(X(f) + X^*(f)\right), \text{ and}$$

$$Y(f) = \mathcal{F}\{\mathrm{Re}\{x(t)\}\} = \frac{1}{2}\left(X(f) + X^*(-f)\right).$$

extra minus sign

(2) When $x(t)$ is real-valued,

$$y(t) = \mathrm{Re}\{x(t)\} = x(t), \text{ and}$$

$$Y(f) = \mathcal{F}\{y(t)\} = \mathcal{F}\{x(t)\} = X(f)$$

Let's check whether $Y(f) = X(f)$ if we use our derived expression for $Y(f)$ above.

Recall that for real-valued $x(t)$,

$$X(-f) = X^*(f)$$

So, $Y(f) = \frac{1}{2}\left(X(f) + X^*(-f)\right) = \frac{1}{2}\left(X(f) + \left(X^*(f)\right)^*\right) = X(f)$. ✓

(3) Let's try another check.

Because $y(t)$ is defined as $\mathrm{Re}\{x(t)\}$,
we know that $y(t)$ will always be real-valued.

Hence, it must also satisfy the conjugate symmetry property:

$$Y(-f) = Y^*(f).$$

So, let's try plugging "$-f$" into our expression for $Y(f)$:

$$Y(f) = \frac{1}{2}\left(X(f) + X^*(-f) \right)$$

This gives

$$Y(-f) = \frac{1}{2}\left(X(-f) + X^*(f) \right)$$

Of course,

$$Y^*(f) = \frac{1}{2}\left(X^*(f) + X(-f) \right)$$

Therefore, $Y(-f) = Y^*(f)$ as expected.

Q7 QAM Key Equation (Complex-Exponential Form)

Monday, November 09, 2015 10:42 AM

(a) $x_b(t) \xrightarrow{\mathcal{F}} X_b(f)$

By the freq. - shift property of Fourier transform,

$$\underbrace{e^{j2\pi f_c t} x_b(t)}_{\text{call this } g(t).} \xrightarrow{\mathcal{F}} X_b(f - f_c)$$

Then, $G(f) = X_b(f - f_c)$ and $x_p(t) = \sqrt{2}\, \mathrm{Re}\{g(t)\}$.

Recall, from the previous problem that $\mathrm{Re}\{g(t)\} \xrightarrow{\mathcal{F}} \frac{1}{2}\left(G(f) + G^*(-f)\right)$.

Hence, $X_p(f) = \sqrt{2} \times \frac{1}{2}\left(G(f) + G^*(-f)\right) = \frac{1}{\sqrt{2}}\left(X_b(f - f_c) + X_b^*(-f - f_c)\right)$

(b) By the freq.-shift property of FT,

$$x_p(t)\, e^{-j2\pi f_c t} \xrightarrow{\mathcal{F}} X_p(f - (-f_c)) = X_p(f + f_c) = \frac{1}{\sqrt{2}}\left(X_b(f + f_c - f_c) + X_b^*(-(f + f_c) - f_c)\right)$$

$$= \frac{1}{\sqrt{2}}\left(X_b(f) + X_b^*(-(f + 2f_c))\right)$$

Therefore, $\sqrt{2}\, x_p(t)\, e^{-j2\pi f_c t} \xrightarrow{\mathcal{F}} X_b(f) + X_b^*(-(f + 2f_c))$

$$\mathrm{LPF}\{\quad\} \xrightarrow{\mathcal{F}} X_b(f) + 0 = X_b(f)$$

LPF